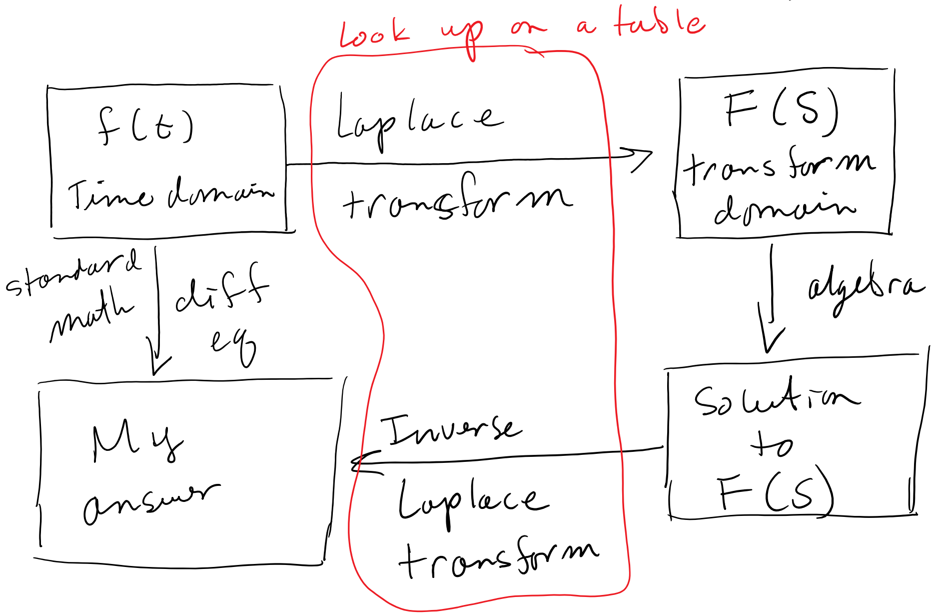
Section 5.1: Laplace Transforms Introduction

Saturday, March 28, 2020 2:05 PM

Two domains

1. Time domain (t)
2. Transform domain (s)



Definition of Laplace Transform

Definition 5.1. Let $f(t)$ be defined for $t \geq 0$ and let s be a real number. Then the **Laplace transform** of $f(t)$, denoted $\mathcal{L}\{f(t)\}$, is the function $F(s)$ defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{5.1}$$

for those values of s for which the improper integral converges.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Ex: Find Laplace transform of $f(t) = \alpha$

$$\mathcal{L}\{\alpha\} = \int_0^\infty \alpha e^{-st}\, dt$$

$u = -st$
$-s\,dt$
$$= -\frac{1}{s} \lim_{A \to \infty} \int_0^A \alpha e^{-st} (-s\,dt)$$

$u = -st$
$du = -s\,dt$

$$= -\frac{1}{s} \lim_{A\to\infty} \int_0^A \alpha e^{-st} (-s\,dt)$$

$$= -\frac{1}{s} \lim_{A\to\infty} \left[\alpha e^{-st} \right]_0^A$$

$$= -\frac{1}{s} \lim_{A\to\infty} \left[\alpha e^{-sA} - \alpha \right]$$

Our answer $\longrightarrow$ If $s > 0$ then $\lim_{A\to\infty} e^{-sA} = 0$

Diverges $\longrightarrow$ If $s < 0$ then $\lim_{A\to\infty} e^{-sA} = \infty$

$$= -\frac{1}{s}(-\alpha) = \boxed{\frac{\alpha}{s} \text{ for } s > 0}$$

$$\mathcal{L}\{8\} = \frac{8}{s}$$

Ex: Find the Laplace transform of $f(t) = t^2$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$$

Table of integrals $\int t^n e^{at}\,dt = \frac{t^n e^{at}}{a} - \frac{n}{a}\int t^{n-1} e^{at}\,dt$

for us $n = 2$ $a = -s$

$$= \frac{t^2 e^{-st}}{-s} \quad \frac{2}{+s} \int_0^\infty t e^{-st}\,dt$$

$$= \frac{t^2 e^{-st}}{-s} + \frac{2}{s}\left[\frac{t e^{-st}}{-s} - \frac{1}{-s}\int_0^\infty e^{-st}\,dt \right]$$

$$= -\frac{t^2 e^{-st}}{s} - \frac{2}{s^2} t e^{-st} + \frac{2}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= -\frac{t^2 e^{-st}}{s} - \frac{2}{s^2}\, t e^{-st} + \frac{2}{s^2}\int_0 e^{-st}\,dt$$

$$= \left(-\frac{t^2}{s} - \frac{2t}{s^2} - \frac{2}{s^3}\right) e^{-st}\Big|_0^{\infty}$$

$s > 0 \quad e^{-\infty} \to 0$

$s < 0 \quad e^{\infty} \to \infty$

$$= 0 - \left(-\frac{2}{s^3}\right)$$

$$\boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3} \qquad s > 0}$$

Ex: $$\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{\alpha t} e^{-st}\,dt$$

$$= \frac{1}{\alpha - s}\int_0^\infty e^{(\alpha - s)t} (\alpha - s)\,dt$$

$u = (\alpha - s)t$

$du = (\alpha - s)\,dt$

$$= \frac{1}{\alpha - s}\left[e^{(\alpha - s)t}\right]_0^\infty$$

Need

$\alpha - s < 0$

$s > \alpha$

$$= \frac{1}{\alpha - s}\left[0 - 1\right]$$

$$\boxed{\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s - \alpha}, \quad s > \alpha}$$

Ex: $$\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{t^2} e^{-st}\,dt$$

$$= \int_0^\infty e^{t^2 - st}\,dt$$

s is a value > 0

$\infty \quad t(t-s)$

$t > s$

$$= \int_0^{\infty} e^{t(t-s)}\,dt$$

$t \to \infty$

$t > s$ then $t - s > 0$

$t(t-s) > 0$ when $t > s$

$\infty \cdot \infty = \infty$

Diverges

$\boxed{e^{t^2} \text{ has no Laplace transform}}$

Two criteria for Laplace transform to exist for $f(t)$.

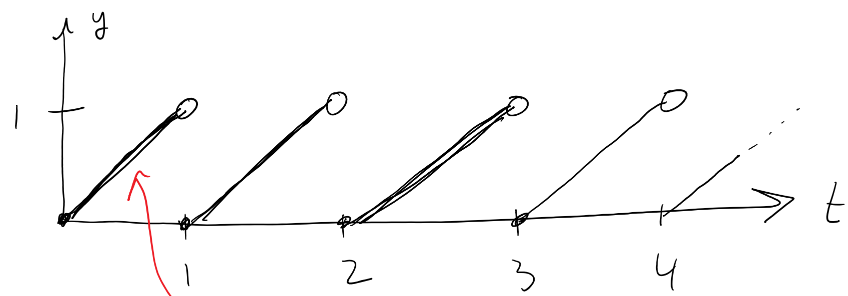
1. $f(t)$ is piecewise continuous.
   a) finite discontinuities in any interval
   b) the limit from the left & right exist at each discontinuity, except at the end pts.

2. $f(t)$ is exponentially bounded

$$|f(t)| < Me^{at}$$

$M > 0$

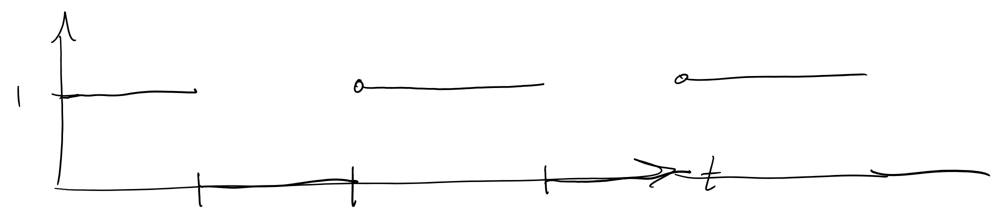
Ex:



$$\boxed{\begin{aligned} f(t) &= t \quad \text{for} \quad 0 \le t < 1 \\ f(t+1) &= f(t) \end{aligned}}$$

periodic with period 1

Ex:



Square wave.

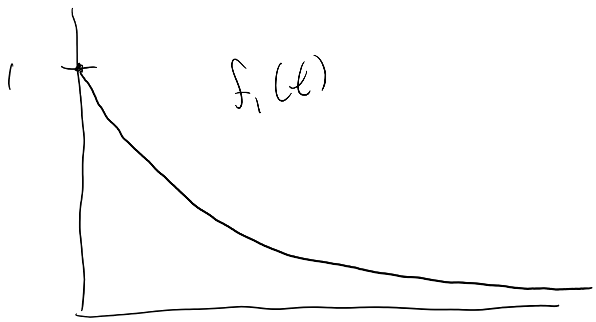
Laplace Rules:

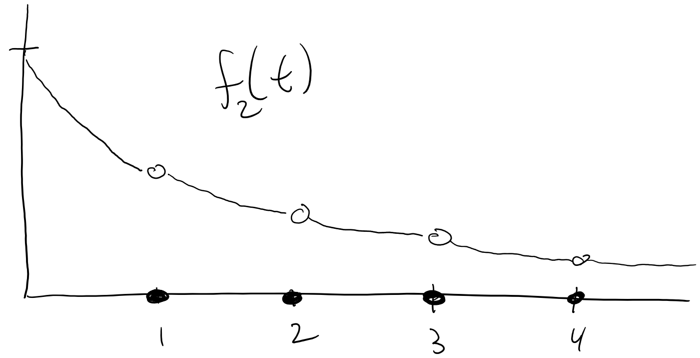
1. Laplace transform is a linear operator

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

2. If $f(t) = f_1(t) \cdot f_2(t)$ & f_1 & f_2 both have Laplace transforms then $\mathcal{L}\{f(t)\}$ exists.

$$f_1(t) = e^{-t} \qquad f_2(t) = \begin{cases} e^{-t} & t \notin \mathbb{Z} \\ 0 & t \in \mathbb{Z} \end{cases}$$





$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\} = \frac{1}{s+1}$$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Ex: Find the inverse Laplace transform of

$$F(s) = \frac{3}{s+2} + \frac{7}{s-2} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

$\alpha = -2$ $\qquad$ $\alpha = +2$

$$\mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{7}{s-2}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 7\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\boxed{= 3e^{-2t} + 7e^{2t}}$$

$$\boxed{= 3e^{-2t} + 7e^{2t}}$$